COVARIANT CORRESPONDENCES AND COVARIANT SETS OF POINTS DEFINED BY A GIVEN CORRESPONDENCE ON AN ALGEBRAIC CURVE

BY

BROOKS JAVINS LOCKHART

A.B., Marshall College, 1937

M.S., West Virginia University, 1940

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1943

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I. INTRODUCTION

Valence correspondences on an algebraic curve C have been the subject of considerable investigation. The most important result has been the Cayley-Brill correspondence principle and the applications which can be made of this principle[1]. A given correspondence T may define other correspondences, or certain sets of points, whose relation to T is independent of birational transformation of C. Such other correspondences or such sets of points will be called covariants of T. Considerable attention will be given to the interpretation of syzygetic relations between a correspondence and its covariants. Certain types of covariant sets of points are defined and illustrated and a considerable portion of the work is devoted to enumeration problems connected with these covariant sets.

Consider an algebraic curve C defined by $f(x_1, x_2, x_3)=0$ or $f(y_1, y_2, y_3)=0$. If by some means a unique set Y of β points is related to any given point (x), this relationship is called a correspondence on C. If there exists an integer γ (positive, zero or negative) such that for any pair of points (x) and (x'), $Y+\gamma.x \equiv Y'+\gamma.x'$, that is, if the sets $Y+\gamma.x$ all belong to the same complete linear series $g_n^{\beta+\gamma}$ on C, then the correspondence is called a valence correspondence and is said to have the valence γ. When the valence is equal to or greater than zero the correspondence is completely defined by a single integral relation $F(x; y)=0$. When a fixed point $y=a$ of C is substituted, this becomes the equation of a curve in (x) and cuts out on C a set X containing say α points. If we

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(1) The Bulletin of the National Research Council, Nr. 63, Selected Topics in Algebraic Geometry, Ch. 7, surveys the literature and most important results.

relate the set X to the point (y) the correspondence so defined is called the inverse of the original correspondence. The inverse has the same valence as the direct correspondence. The symbol $T(\alpha,\beta,\gamma)$ is used to indicate a correspondence T of valence γ having β points in the direct and α points in the inverse correspondence.

The sum of two correspondences $T_1(\alpha_1,\beta_1,\gamma_1)$ and $T_2(\alpha_2,\beta_2,\gamma_2)$ relates to a point (x) the two sets Y_1+Y_2 which are obtained by T_1 and T_2. The indices of the sum are $(\alpha_1+\alpha_2,\ \beta_1+\beta_2,\ \gamma_1+\gamma_2)$. Addition of correspondences is commutative and associative. Multiplication of two correspondences is defined as the successive application of T_1 and T_2. If $x \xrightarrow{T_1} Y_1 = y_1, y_2, \cdots, y_\beta$, and $y_i \xrightarrow{T_2} Y_{i,2}$, then $x \xrightarrow{T_1 T_2} \sum_{i=1}^{i=\beta_1} Y_{i,2}$. By definition of valence $Y_1+\gamma_1 x_1 \equiv Y_1'+\gamma_1 x'$; $Y_{i,2}+\gamma_2 y_i \equiv Y_{i,2}'+\gamma_2 y_i'$ or $\sum_{i=1}^{i=\beta_1} Y_{i,2}+\gamma_2 Y_1 \equiv \sum_{i=1}^{i=\beta_1} Y_{i,2}'+\gamma_2 Y_1'$. This leads to

$$\sum_{i=1}^{i=\beta_1} Y_{i,2}-\gamma_1\gamma_2 x \equiv \sum_{i=1}^{i=\beta_1} Y_{i,2}'-\gamma_1\gamma_2 x'$$

Hence $T_1 T_2$ has the valence $-\gamma_1\gamma_2$ and the indices $(\alpha_1\alpha_2,\beta_1\beta_2,-\gamma_1\gamma_2)$. Multiplication is non-commutative but it is associative and the distributive laws with respect to addition hold.

Let $T(\alpha,\beta,-\gamma)$ have negative valence. Then T is completely defined by two integral relations $F_1(x,y)=0$; $F_2(x,y)=0$. For, it is clear that one function is not enough since the equivalence $Y-\gamma x \equiv Y'-\gamma x'$ implied by the valence $-\gamma$ is an equivalence of virtual sets. Construct a correspondence $T_1(\alpha_1,\beta_1,\gamma)$ as the sum of γ elementary correspondences. Then $T+T_1$ has valence 0 and is determined by a single function $F_1(x,y)=0$. If a second correspondence $T_2(\alpha_2,\beta_2,\gamma)$ is constructed as the sum of γ elementary correspondences distinct from those used for T_1, $T+T_2$ has valence 0 and is determined by $F_2(x,y)=0$. Now F_1 and F_2 have in common on C exactly the sets X and Y which

correspond to (a) by T^{-1} and T. . Hence these two functions are sufficient to determine T .

The correspondence which relates each point (x) of the curve to itself will be of particular interest. Clearly its indices are $\alpha=\beta=1$. It can be shown to have valence $\gamma=-1$. Since it behaves as an identity under multiplication it is suitable to indicate this correspondence by $I(1,1,-1)$.

II. COVARIANT CORRESPONDENCES

The notion of a valence correspondence is itself invariant under birational transformations of the curve since it is completely determined by algebraic relations which are themselves invariant. The inverse, T^{-1}, is the simplest covariant of a given correspondence, $T(\alpha,\beta,\gamma)$. By the multiplication formula the product TT^{-1} should have the indices $(\alpha\beta, \alpha\beta, -\gamma^2)$. But the identity will be included β times in the set corresponding to (x) under TT^{-1}. For, if $a \xrightarrow{T} a_1, a_2, \dots, a_\beta$ then by definition of the inverse $a_i \xrightarrow{T^{-1}} a_{i1}=a, a_{i2}, \dots, a_{i\alpha}$. So the product TT^{-1} contains $\beta\cdot I$ as a degenerate part and the proper part of the product has the indices $(\alpha(\beta-1), \alpha(\beta-1), -\gamma^2+\beta)$. Thus the following relation holds: $TT^{-1} = \beta\cdot I + \overline{TT^{-1}}(\alpha(\beta-1), \alpha(\beta-1), -\gamma^2+\beta)$ where TT^{-1} indicates the formal product and $\overline{TT^{-1}}$ indicates the proper part of the product. By the way in which they are defined TT^{-1} and $\overline{TT^{-1}}$ are clearly covariants of T . Such a relation between the covariants TT^{-1} and $\overline{TT^{-1}}$ is of the same character as a syzygy among algebraic covariants. A similar relation is $T^{-1}T = \alpha\cdot I + \overline{T^{-1}T}$.

Suppose $T_1(\alpha,\beta,\gamma)$ and $T_2(\beta,\alpha,\gamma)$ are two correspondences satisfying the relation $T_1T_2=\beta\cdot I + \overline{T_1T_2}$. If $a \xrightarrow{T_1^{-1}} a_1, a_2, \dots, a_\alpha, \dots\dots\dots a_\alpha$ then $a_i \xrightarrow{T_1} a_{i1}=a, a_{i2}, \dots, a_{i\beta}$.

By the relation assumed for T_1, T_2 it follows that $a_{i1} = a \xrightarrow{T_2} a_{i11} = a_i, a_{i12}, \dots\dots, a_{i1\alpha}$. By repeated argument $a \xrightarrow{T_2} a_1, a_2, \dots\dots, a_i, \dots\dots, a_\alpha$. Hence $T_2 = T_1^{-1}$. Therefore the following theorem holds.

Theorem: Two correspondences $T_1(\alpha, \beta, \gamma)$, $T_2(\beta, \alpha, \gamma)$ having the proper indices are the inverses of one another if and only if they satisfy the syzygetic relations $T_1 T_2 = \beta \cdot I + \overline{T_1 T_2}$ and $T_2 T_1 = \alpha \cdot I + \overline{T_2 T_1}$. These two relations are interchangeable since one implies the other.

A correspondence S will be called symmetric if $S = S^{-1}$. This implies at once that the indices are $S(\alpha, \alpha, \gamma)$. By the preceding theorem $S^2 = S \cdot S^{-1} = \alpha \cdot I + \overline{S^2}(\alpha^2 - \alpha, \alpha^2 - \alpha, -\gamma^2 + \alpha)$. Conversely, suppose $S(\alpha, \alpha, \gamma)$ is a correspondence which satisfies this relation. Then the product $S \cdot S = S^2$ satisfies the preceding condition for two inverse correspondences. Hence the following

Theorem: A correspondence $S(\alpha, \alpha, \gamma)$ is symmetric if and only if it satisfies the covariant relation

$$S^2 = \alpha \cdot I + \overline{S^2}(\alpha^2 - \alpha, \alpha^2 - \alpha, -\gamma^2 + \alpha).$$

Higher powers of S also degenerate so that the preceding relation leads to a whole system of syzygetic relations. It is easy to show that the proper powers of the symmetric correspondence S satisfy the relation $\overline{S^k} \cdot S = (\alpha - 1) \cdot \overline{S^{k-1}} + \overline{S^{k+1}}$. Hence the proper n-th power $\overline{S^n}$ has the indices $\alpha(\alpha-1)^{n-1}$, $\alpha(\alpha-1)^{n-1}$. There seems to be no simple way to express the valence, $\nu(\overline{S^n})$, of $\overline{S^n}$ in terms of α and γ. But it is clear from what precedes that $\nu(\overline{S^n})$ satisfies the recursion formula

$$\nu(\overline{S^n}) = -\gamma \cdot \nu(\overline{S^{n-1}}) - (\alpha - 1) \cdot \nu(\overline{S^{n-2}}).$$

An elementary correspondence is one which is determined by a pencil of curves. Such a correspondence is symmetric

and has valence $\nu = 1$. A necessary and sufficient condition for an elementary correspondence is given by the following

Theorem: A symmetric correspondence $S(\alpha, \alpha, 1)$ is an elementary correspondence if and only if it satisfies the covariant relation $S^2 = (\alpha - 1) \cdot S + \alpha \cdot I$.

Any correspondence $T(\alpha, \beta, \gamma)$ defines the symmetric covariant $\overline{TT^{-1}}$. Any symmetric correspondence $S(\beta(\alpha-1), \beta(\alpha-1), -\gamma^2+\beta)$ which has the form $S = \overline{TT^{-1}}$ satisfies the covariant relation $S^2 = \beta(\alpha-1) \cdot I + (\alpha-2) \cdot S + \overline{S^2}$.

III. COVARIANT SETS OF POINTS

Let (a) be a point on the curve C . Applying the correspondence $T(\alpha, \beta, \gamma)$ repeatedly gives a sequence of points $a \xrightarrow{T} b \xrightarrow{T} c \xrightarrow{T} d$ and so on. When (a) is generically chosen this sequence is infinite, that is, the point (a) does not occur again in the sequence. A finite sequence of m points $(a_1, a_2, \dots\dots, a_m)$ having the property that $a_i \xrightarrow{T} a_{i+1}$ for any $i < m$ and $a_m \xrightarrow{T} a_1$ will be called a cyclic m-ad of the correspondence T . The aggregate of cyclic m-ads will be called a covariant set of T .

The simplest cyclic m-ads are those with one element, that is, a point (a) such that $a \xrightarrow{T} a$. The Cayley-Brill correspondence principle states that the number of such fixed points of $T(\alpha, \beta, \gamma)$ is $u = \alpha + \beta + 2p\gamma$ where p represents the genus of the curve. A cyclic pair (a, b) such that $a \xrightarrow{T} b$, $b \xrightarrow{T} a$ is suitably called an involutorial pair of T . If T is symmetric there will be an infinite number of involutorial pairs. For, if (a) is any point on the curve and $a \xrightarrow{T} b$ then by definition $b \xrightarrow{T^{-1}} a$ and since $T = T^{-1}$, any pair of corresponding points will be an involutorial pair.

To determine the number of involutorial pairs for a non-symmetric correspondence note that both points (a) and (b) of such a pair will be fixed points of T^2. Not every fixed point of T^2 will arise from involutorial pairs since clearly fixed points of T will also be fixed under T^2. Then it would be expected that the fixed points of T enter into the fixed points of T^2 a certain fixed number of times. To determine that this occurs only once is fairly simple. Hence the number μ_2 of involutorial pairs of $T(\alpha,\beta,\gamma)$ is given by $2\mu_2 = u_{T^2} - u_T = \alpha(\alpha-1) + \beta(\beta-1) - 2\gamma(\gamma-1)$.

For the symmetric correspondences, which have an infinite number of cyclic pairs, the formulas for the number of cyclic n-ads will be very different from those for non-symmetric correspondences. The question naturally arises as to the possibility of the existence of valence correspondences having an infinite number of cyclic n-ads, $n>2$. That there are one-to-one transformations which are of period n is well known. These periodic correspondences are all singular. Since the only known examples are singular it seems probable that any algebraic correspondence which admits an infinite number of cyclic n-ads ($n>2$) is necessarily singular. No attempt will be made to prove this statement here but we will simply assume that the correspondences under consideration have no such infinite sets.

The number of cyclic n-ads for a general correspondence $T(\alpha,\beta,\gamma)$ will be determined by the number of coincidences of T^n. Reductions must be made for the sets of repeated points which would be counted in the coincidences u_n of T^n. For any divisor n_i of n the sets of n_i points repeated n/n_i times make improper cyclic n-ads. Hence the following

Theorem: For a general non-symmetric correspondence $T(\alpha,\beta,\gamma)$ the number μ_n of cyclic n-ads is given by

$$n\mu_n = u_n - \sum n_i \mu_{n_i}$$

where the summation ranges over all divisors n_i of n.

The values of μ_n given by this formula are all integers. This is proved by induction with respect to the number of prime divisors of n.

Suppose $S(\alpha,\alpha,\gamma)$ is a general symmetric correspondence, satisfying only the covariant relation $S^2 = \alpha\cdot I + \overline{S^2}$. The number of cyclic n-ads will be determined by the coincidences u_n of $\overline{S^n}$. The types of improper sets which are enumerated in u_n will be illustrated when $n=5$. If (a,b,c) is a cyclic triad and (d) is one of the $(\alpha-2)$ other points such that $d \xrightarrow{S} a$, the improper coincidences $d \xrightarrow{S} a \xrightarrow{S} b \xrightarrow{S} c \xrightarrow{S} a \xrightarrow{S} d$ and $b \xrightarrow{S} a \xrightarrow{S} c \xrightarrow{S} b \xrightarrow{S} a \xrightarrow{S} d$ occur. There are $6(\alpha-2)\mu_3$ of this type. If (a) is a coincidence of S and (b) is one of the $(\alpha-1)^2$ points such that $b \xrightarrow{\overline{S^2}} a$, the improper coincidence $b \xrightarrow{\overline{S^2}} a \xrightarrow{S} a \xrightarrow{\overline{S^2}} b$ occurs. There are $(\alpha-1)^2 u_1$ of this type. Hence $10\mu_5 = u_5 - 6(\alpha-2)\mu_3 - (\alpha-1)^2 u_1 = u_5 - (\alpha-2)u_3 - (\alpha+1)u_1$. Further consideration leads to the following general formulas.

A. $n = 2m+1$ is odd.

$$2n\mu_n = u_n - 2(n-2)(\alpha-2)\mu_{n-2} - 2(n-4)(\alpha-2)(\alpha+1)\mu_{n-4} - \cdots\cdots\cdots$$
$$- 2(n-2k)(\alpha-2)(\alpha-1)^{k-1}\mu_{n-2k} - \cdots\cdots\cdots\cdots\cdots$$
$$- 2\cdot 3(\alpha-2)(\alpha-1)^{m-2}\mu_3 - (\alpha-1)^m u_1 - \sum 2n_i\mu_{n_i}$$

B. $n = 2m$ is even.

$$2n\mu_n = u_n - 2(n-2)(\alpha-2)\mu_{n-2} - 2(n-4)(\alpha-2)(\alpha-1)\mu_{n-4} - \cdots\cdots$$
$$- 2(n-2k)(\alpha-2)(\alpha-1)^{k-1}\mu_{n-2k} - \cdots\cdots\cdots\cdots$$
$$- 2\cdot 4(\alpha-2)(\alpha-1)^{m-3}\mu_4 - (\alpha-1)^{m-1}\mu_2 - \sum 2n_i\mu_{n_i}$$

where the summations vary over all divisors $n_i > 2$ of n and where it is agreed to omit the terms involving divisors

of $n-2, n-4, \cdots\cdots\cdots$ in the expressions for $\mu_{n-2}, \mu_{n-4}, \cdots\cdots\cdots$.

By induction these formulas are proved to reduce to the form given in the following

Theorem: The number μ_n of cyclic n-ads for the symmetric correspondence $S(\alpha, \alpha, \gamma)$ is given by the formulas

n odd: $2n\mu_n = u_n - (\alpha-2)(u_{n-2} + u_{n-4} + \cdots\cdots + u_5 + u_3) - (\alpha-1)u_1 - \sum 2n_i \mu_{n_i}$

n even: $2n\mu_n = u_n - (\alpha-2)(u_{n-2} + u_{n-4} + \cdots\cdots + u_6 + u_4) - (\alpha-1)u_2 - \sum 2n_i \mu_{n_i}$

where u_k denotes the number of coincidences of $\overline{S^k}$ as given by the Cayley-Brill principle and the summations range over all divisors $n_i > 2$ of n.

On the quartic curve of genus 3 the correspondence by which two points (a) and (b) correspond to each other when they lie on the same tangent to the curve is $S(10, 10, 6)$ Its cyclic n-ads are given by the preceding theorem. A proper cyclic n-ad for S constitutes an n-sided polygon which is inscribed in and circumscribed about the curve. Such a set of lines is a degenerate contact curve of order n. Hence the preceding theory can be applied to finding the number of contact curves of the quartic which break up into lines. Typical results obtained are as follows: there are 8 contact cubics, 336 contact quintics and $19,032$ contact curves of order 7 in each of the 36 systems which degenerate into lines. In the other 28 systems of contact curves of odd order degeneration is not possible. There are 24 contact quartics and $1,392$ contact sextics in each of the 63 systems which degenerate into lines.

VITA

Brooks Javins Lockhart was born in Sandyville, West Virginia, February 8, 1920, and received his elementary and secondary education there. He entered Marshall College in September, 1933, and received the degree of Bachelor of Arts (cum laude) from that institution in June, 1937. After working in the office of the State Auditor of West Virginia until June, 1938, he entered the Graduate School of West Virginia University. He studied there until February, 1940, when he received the degree of Master of Science. During the succeeding semester he taught mathematics in Man High School, Man, West Virginia. Since September, 1940, he has been a graduate assistant in the department of mathematics at the University of Illinois.

He is a member of Kappa Delta Pi, Chi Beta Phi and the American Mathematical Society.